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A BOUNDARY-LAYER METHOD IN POROUS BODY HEAT AND MASS TRANSFER

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Abstract—Boundary-layer technique developed by Goodman in heat-transfer problems has been extended to coupled phenomena of heat and mass transfer in porous medium. To justify the application of the method in heat- and mass-transfer problems, a linear problem with boundary conditions of second and third kind respectively has been discussed and results compared with already known exact solutions. A non-linear problem where the Luikov number is taken as linearly dependent on temperature and mass-transfer potential, has also been discussed and some results have been exhibited graphically.

NOMENCLATURE

- x, length coordinate;
- L, thickness of the body in problem;
- t, temperature;
- t_c, temperature of the surrounding atmosphere;
- θ, mass-transfer potential;
- θ_p, equilibrium value of mass-transfer potential;
- τ, time;
- λ_q, thermal conductivity;
- λ_m, mass conductivity;
- a_q, thermal diffusivity;
- a_m, moisture diffusivity;
- c_m, specific mass capacity;
- c_q, specific heat capacity;
- γ_0, density of porous skeleton;
- δ_s, Soret coefficient;
- ϵ, coefficient of internal evaporation;
- ρ, specific heat of evaporation;
- α, surface heat-transfer coefficient;
- q_m, mass flux per unit area;
- q_q, heat flux per unit area;
- X, non-dimensional length $(= x/L)$;
- Lu, Luikov number $(= a_m/a_q)$;
- Ko, Kossovitch number $(= \rho c_m \Delta\theta/c_q \Delta t)$;
- Pn, Posnov number $(= \delta_s \Delta t/\Delta\theta)$;
- Ki_m, Kirpichev number for mass transfer $(= q_m t/\lambda_m \Delta\theta)$;
- Ki_q, Kirpichev number for heat transfer $(= q_q L/\lambda_q \Delta t)$;
- Bi_q, Biot number for heat transfer $(= \alpha L/\lambda_q)$;
- Fo, Fourier number $(= a_q \tau/L^2)$;
- T, non-dimensional temperature $(= t - t_0/t_c - t_0)$;
- Θ, non-dimensional mass-transfer potential $(= \theta_0 - \theta/\theta_0 - \theta_p)$.

INTRODUCTION

RECENTLY Goodman has applied a technique known as the "Heat Balance Integral Method" to solve some linear and non-linear problems in heat transfer. In this paper we extend the above technique to the solution of certain problems in coupled phenomena of heat and mass transfer in porous media. Neglecting the convective molar transfer of mass and heat these equations are

$$\begin{aligned}\frac{\partial t}{\partial \tau} &= a_q \nabla^2 t + \frac{\epsilon \rho c_m}{c_q}\frac{\partial \theta}{\partial \tau}\\ \frac{\partial \theta}{\partial \tau} &= a_m \nabla^2 \theta + a_m \delta_s \nabla^2 t.\end{aligned} \tag{1}$$

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In general these equations are non-linear because of dependence of a_m, a_q, ϵ, δ_s, etc., on temperature and moisture concentration.

However, for many practical applications, calculations are carried out by taking these coefficients to be constant by dividing the entire range of variation of these coefficients into various sub-ranges and solving the above equations with average values of these coefficients, thus making these equations linear. This is, however, a necessity of a simple approximate method to give useful analytical results for the equations where the variation of transfer coefficients with temperature and moisture transfer potential is taken into account.

In the application of heat balance integral to heat-transfer problems for plates of finite thickness it is assumed that a thermal layer analogous to the velocity boundary layer exists whose thickness grows with time. The thickness of this layer is specified by the surface where the conditions of zero heat transfer and equality of temperature to its initial value are satisfied. Therefore, as long as the thickness of the thermal layer is less than the thickness of the plate it behaves as an infinite medium, as the boundary condition on the other end of the plate does not matter. At the transition time when the thickness of the thermal layer is just equal to the plate thickness, the boundary condition on the other end comes into play. The solution of the problem is thus split up into two parts—one valid in the range $0 < \tau < \tau'$ (where τ' is the transition time) and the other for higher values of time (i.e. $\tau > \tau'$).

In the case of coupled phenomena of heat and mass transfer governed by the equations (1), the extension of Goodman's technique envisages the specification of the relative rate of heat- and mass-transfer processes, i.e. it is essential to differentiate the cases where heat transfer precedes mass transfer in the very initial stage of the process or vice versa. It has been observed that such an assumption regarding the relative rate of the progress of the two layers, energy penetration depth and mass penetration depth give a relation between the transfer coefficients.

If the exact equations (1) are taken into account right from the beginning it becomes difficult to obtain explicit relations between the transfer coefficients. In a large number of applications it is known that the term δ_s is small and is completely neglected in the above equations. However, in the approach followed here δ_s is neglected for the first stage (up till the transition time, which is small for moderately thin plates) and complete equations of heat and mass transfer are solved in the second stage of the process. In the first stage, if heat transfer is assumed to lag behind the mass transfer we find that it envisages a relation

$$Lu > \frac{1}{1 + \epsilon Ko \dfrac{Ki_m}{Ki_q}}$$

for boundary conditions of second kind,

or

$$Lu > \frac{1}{1 + \epsilon Ko}$$

for boundary conditions of first kind.

The above inequalities are reversed if the mass transfer lags behind the heat transfer in the first stage. These inequalities seem to modify the crieterion of Luikov (discussed on p. 173 of reference [1])

$$Lu \gtrless 1.$$

In the second stage the procedure followed by us is similar to that of Goodman.

In contrast to Goodman's method in heat transfer, in the case of combined heat and mass transfer we have to integrate both the heat- and mass-transfer equations leading to the "Energy Balance Integral" and "Mass Balance Integral". The solutions then satisfy both heat- and mass-transfer equations on an average and therefore allow representation of heat- and mass-transfer potential in the form of polynomials in the space variable with time-dependent coefficients.

To illustrate the method, we first discuss the case of heat and mass transfer in an infinitely long slab of finite width under the boundary conditions of the second kind, completely neglecting δ_s. The results are obtained for both the cases, i.e. when heat transfer precedes mass transfer and vice versa.

Secondly we discuss the problem of heat and mass transfer in an infinite plate (finite width) with the boundary conditions of the third kind. In this case, in the first phase we neglect the thermal diffusion term while in the second phase the complete equations have been taken into account. The penetration depth (for heat transfer) has been obtained in an implicit form for the case when heat transfer precedes mass transfer.

Finally we have discussed a non-linear problem where the diffusivities of heat and mass transfer have been assumed to vary linearly with temperature and mass transfer potential. This non-linear problem has been studied with the boundary conditions of the first kind at one end. The solution has been obtained for both the situations, i.e. when heat transfer precedes mass transfer and vice versa and the results have been graphically depicted.

1. APPLICATION OF THE METHOD TO SOLUTION OF A PROBLEM WITH BOUNDARY CONDITIONS OF THE SECOND KIND

Problem. An infinite porous plate of finite thickness is initially at temperature t_0 and mass-transfer potential θ_0. One face of the plate ($x = 0$) is insulated to heat and mass transfer while the other face ($x = L$) is maintained at constant heat and mass transfer flux. Neglecting the thermal diffusion determine the temperature and mass transfer potential distributions inside the plate.

The differential equations governing the process of heat and mass-transfer together with initial and boundary conditions appropriate to the above problem are as under:

$$\frac{\partial t}{\partial \tau} = a_q \frac{\partial^2 t}{\partial x^2} + \frac{\epsilon \rho c_m}{c_q}\frac{\partial \theta}{\partial \tau} \tag{1.1}$$

$$\frac{\partial \theta}{\partial \tau} = a_m \frac{\partial^2 \theta}{\partial x^2} \qquad 0 < x < L \qquad \tau > 0 \tag{1.2}$$

$$\left.\begin{aligned} t &= t_0 \quad (1.3)\\ \theta &= \theta_0 \quad (1.4)\end{aligned}\right] \quad 0 \leqslant x \leqslant L \qquad \tau = 0$$

$$\left.\begin{aligned} -\lambda_q \frac{\partial t}{\partial x} + q_q &= 0 \quad (1.5)\\ \lambda_m \frac{\partial \theta}{\partial x} + q_m &= 0 \quad (1.6)\end{aligned}\right] \quad x = L \qquad \tau > 0$$

$$\left.\begin{aligned} \frac{\partial t}{\partial x} &= 0 \quad (1.7)\\ \frac{\partial \theta}{\partial x} &= 0 \quad (1.8)\end{aligned}\right] \quad x = 0 \qquad \tau > 0.$$

The equations (1.1), (1.2) can be written as

$$\frac{\partial t}{\partial \tau} = K_{11}\frac{\partial^2 t}{\partial x^2} + K_{12}\frac{\partial^2 \theta}{\partial x^2} \tag{1.9}$$

$$\frac{\partial \theta}{\partial \tau} = K_{21}\frac{\partial^2 \theta}{\partial x^2} \qquad 0 < x < L \qquad \tau > 0 \tag{1.10}$$

where

$$K_{11} = a_q$$

$$K_{12} = \frac{\epsilon\rho}{c_q} c_m a_m$$

$$K_{21} = a_m.$$

SOLUTION

First phase

We assume that at any time τ the temperature and mass disturbances have penetrated inside the plate up to the distances $\delta'(\tau)$, $\delta(\tau)$ respectively measured from $x = L$. The concentration and temperature beyond the penetration depths remain at initial values, i.e. the conditions at $x = L - \delta'$ and $x = L - \delta$ are

$$\left.\begin{aligned} t &= t_0 \quad (1.11)\\ \frac{\partial t}{\partial x} &= 0 \quad (1.12)\end{aligned}\right] \quad x = L - \delta'$$

$$\left.\begin{aligned} \theta &= \theta_0 \quad (1.13) \\ \frac{\partial \theta}{\partial x} &= 0 \quad (1.14) \end{aligned}\right] x = L - \delta.$$

Integrating first the mass-transfer equation (1.10) from $x = L$ to $x = L - \delta$ and using (1.14) we obtain the mass balance integral as

$$\int_L^{L-\delta} \frac{\partial \theta}{\partial \tau} \mathrm{d}x = -K_{21} \left.\frac{\partial \theta}{\partial x}\right|_{x=L}. \quad (1.15)$$

Assume a parabolic polynomial profile for mass-transfer potential as

$$\theta = A_0 + A_1(L - x) + A_2(L - x)^2 \quad (1.16)$$

where A_0, A_1, A_2 are functions of time.

The coefficients A_0, A_1, A_2 can be determined from three boundary conditions (1.13), (1.14) and (1.6) and then θ can be written as

$$\theta = \theta_0 - \frac{q_m}{2\lambda_m \delta}(\delta - L + x)^2. \quad (1.17)$$

as a function of time can now be determined by putting θ from (1.17) into mass balance integral (1.15).

Thus we get a first-order differential equation in the form

$$\frac{\mathrm{d}(\delta^2)}{\mathrm{d}\tau} = 6K_{21}. \quad (1.18)$$

The above equation when integrated gives

$$\delta(\tau) = \sqrt{(6K_{21}\tau)} \quad (1.19)$$

where $\delta(0) = 0$ is the initial condition.

Having determined δ we obtain θ from equation (1.17).

We assume similarly a parabolic profile for temperature as

$$t = B_0 + B_1(L - x) + B_2(L - x)^2. \quad (1.20)$$

On integrating the heat-transfer equation (1.9) from $x = L$ to $x = L - \delta'$ we have to consider two cases namely $\delta' > \delta$ or $\delta' < \delta$.

When $\delta' > \delta$. (i.e. when heat transfer precedes mass transfer.) In this case the heat balance integral under the conditions (1.12) and $\partial\theta/\partial x|_{x=L-\delta'} = 0$ can be written as

$$\int_L^{L-\delta'} \frac{\partial t}{\partial \tau} \mathrm{d}x = -K_{11} \left.\frac{\partial t}{\partial x}\right|_{x=L} - K_{12} \left.\frac{\partial \theta}{\partial x}\right|_{x=L}. \quad (1.21)$$

The polynomial for temperature t (1.20) can be determined in terms of δ' in the first phase and $\delta' > \delta$ from conditions (1.5), (1.11), (1.12) and can be written as

$$t = t_0 + \frac{q_q}{2\lambda_q \delta'}(\delta' - L + x)^2. \quad (1.22)$$

To determine δ' we substitute (1.22) in (1.21) as we did in the case of mass penetration depth and obtain a 1st order differential equation for δ' which when solved gives

$$\delta'(\tau) = \sqrt{\left[6K_{11}\left(1 - \frac{\epsilon \rho c_m a_m q_m \lambda_q}{c_q a_q q_q \lambda_m}\right)\tau\right]} \quad (1.23)$$

where $\delta'(0) = 0$ is taken as the initial condition.

Substituting δ' from (1.23) in (1.22), temperature profile in the first phase is determined for this case ($\delta' > \delta$).

When $\delta' < \delta$. (i.e. when mass transfer precedes heat transfer.)

In this case the heat balance integral under the conditions (1.5), (1.6) and the value of $\partial\theta/\partial x|_{x=L-\delta'}$ can be written as

$$\int_L^{L-\delta'} \frac{\partial t}{\partial \tau} \mathrm{d}x = -K_{11}\frac{q_q}{\lambda_q} + K_{12}\frac{q_m}{\lambda_m}\frac{\delta'}{\delta}. \quad (1.24)$$

where the value of $\partial\theta/\partial x|_{x=L-\delta'}$ from the mass transfer potential profile (1.17) has been used.

Substituting the value of δ from (1.19) and of t from (1.22) a first-order differential equation for δ' is obtained in the form

$$\frac{\mathrm{d}(\delta'^2)}{\mathrm{d}\tau} = A - \frac{B\delta'}{\sqrt{\tau}} \quad (1.25)$$

where

$$A = 6K_{21} \quad (1.26)$$

$$B = \frac{K_{12}}{\sqrt{(6K_{21})}} \frac{q_m \lambda_q}{\lambda_m a_q}. \tag{1.27}$$

The solution of (1.25) making use of the initial condition $\delta'(0) = 0$ is

$$\delta'(\tau) = \left[\sqrt{\left(A + \frac{B^2}{4}\right)} - \frac{B}{2}\right]\sqrt{\tau}. \tag{1.28}$$

Having determined $\delta'(\tau)$ the temperature profile in this case can be determined from (1.22) by substituting the value of δ' from (1.28) into (1.22)

The inequalities $\delta' \gtrless \delta$ when expressed in terms of transfer coefficients become respectively

$$Lu \gtrless \frac{1}{1 + \epsilon Ko(Ki_m/Ki_q)}.$$

We have discussed two cases ($\delta' \gtrless \delta$) in the first phase and have determined the distribution for temperature in each case. We shall now pass on to the second phase and shall determine the final distributions of temperature in each case. It has however to be noted that the two situations are only relevant to temperature distributions, mass-transfer equation being independent.

Second phase

The idea of penetration distance ceases to be valid when the energy or mass penetration depth reaches the other face of the plate and we have to take into account the boundary conditions at the other face, and the profiles have to be redetermined to include the effect of this boundary. Moreover, two conditions which are satisfied at the end of the mass or energy layer are replaced only by one boundary condition at this face ($x = L$) and out of three constants for the parabolic profile only two can be determined, the third has to be determined from the balance integral. However, if higher polynomials are used, the other constants can be determined from some derived conditions and the differential equations.

The initial distributions for the second phase can be obtained from first phase distributions by putting δ' or $\delta = L$.

When $\delta' > \delta$. The initial distributions of temperature and mass transfer potential are

$$t = t_0 + \frac{q_q}{2\lambda_q^4} x^2 \quad \tau = \tau_1 \tag{1.29}$$

$$\theta = \theta_0 - \frac{q_m}{2\lambda_m^4} x^2 \quad \tau = \tau_2 \tag{1.30}$$

where

$$\tau_1 = \frac{L^2}{6K_{11}\left[1 - \frac{\epsilon\rho a_m c_m \lambda_q}{c_q a_q \lambda_m}\right]} \tag{1.31}$$

and

$$\tau_2 = \frac{L^2}{6K_{21}}. \tag{1.32}$$

We assume the parabolic profiles for the second phase as

$$t = A_0' + A_1'x + A_2'x^2 \tag{1.33}$$

$$\theta = B_0' + B_1'x + B_2'x^2. \tag{1.34}$$

Making use of the boundary conditions (1.5), (1.7), (1.33) can be written as

$$t = A_0' + \frac{q_q}{2\lambda_q L} x^2. \tag{1.35}$$

Integrating now the heat-transfer equation (1.1) with respect to x from $x = 0$ to $x = L$ we obtain heat balance integral as

$$\frac{\partial}{\partial\tau}\int_0^L t\,\mathrm{d}x = K_{11}\frac{q_q}{\lambda_q} - K_{12}\frac{q_m}{\lambda_m}. \tag{1.36}$$

Substituting t from (1.35) into (1.36) we get the first order differential equation for A_0' as

$$\frac{\mathrm{d}A_0'}{\mathrm{d}\tau} = K_{11}\frac{q_q}{\lambda_q L} - K_{12}\frac{q_m}{\lambda_m L} \tag{1.37}$$

which determines A_0' using $A_0'(\tau_1) = t_0$.

Substituting A_0' in (1.35) we get the final distribution for temperature as

$$\frac{t - t_0}{t_c - t_0} = \frac{K_{11} q_q}{\lambda_q L(t_c - t_0)} \left[1 - \frac{\epsilon \rho c_m a_m q_m \lambda_q}{c_q a_q q_q \lambda_m} \right] \times (\tau - \tau_1) + \frac{q_q x^2}{2\lambda_q L(t_c - t_0)} \quad (1.38)$$

which can be written in the non-dimensional form as

$$T = Ki_q[Fo - \tfrac{1}{6}(1 - 3X^2)] - \epsilon Lu Ko Ki_m Fo \quad (1.39)$$

where

$$Ki_m = \frac{q_m L}{\lambda_m(\theta_0 - \theta_p)}$$

$$Ki_q = \frac{q_q L}{\lambda_q(t_c - t_0)}$$

$$Ko = \frac{\rho c_m}{c_q} \frac{\theta_0 - \theta_p}{t_c - t_0}.$$

Proceeding exactly in the same manner we get

$$\theta = \theta_0 - \frac{K_{21} q_m}{\lambda_m L}(\tau - \tau_2) - \frac{1}{2L} \frac{q_m}{\lambda_m} x^2 \quad (1.40)$$

where we have made use of $B_0'(\tau_2) = \theta_0$ from (1.30) and τ_2 is given by (1.32).

The non-dimensional form of (1.40) is

$$\Theta = Ki_m[LuFo - \tfrac{1}{6}(1 - 3X^2)]. \quad (1.41)$$

When $\delta' < \delta$. The distribution of temperature in this case is similar to that for the case $\delta' > \delta$, i.e. it is similar to (1.38) except that τ_1 has to be replaced by τ_3 which can be found from (1.28) by putting $\delta' = L$ and then the distributions of temperature in non-dimensional form is

$$T = Fo[Ki_q - \epsilon Lu Ko Ki_m] - Ki_q[Fo_3 - \tfrac{1}{2}X^2] \quad (1.42)$$

where Fo_3 is the non-dimensional form of the transit time τ_3, i.e.

$$Fo_3 = a_q \tau_3 / L^2. \quad (1.43)$$

Unlike the temperature distribution the mass-transfer potential distribution will be the same as in the $\delta' > \delta$ case on account of independence of mass-transfer equation of the heat-transfer equation.

The temperature distributions (1.39) and (1.42) are true under the inequalities

$$Lu < \frac{1}{1 + \epsilon Ko Ki_m / Ki_q}$$

and

$$Lu > \frac{1}{1 + \epsilon Ko Ki_m / Ki_q}$$

respectively, which are the manifestations of the inequalities $\delta' > \delta$ and $\delta' < \delta$. The mass-transfer potential solution (1.41) is however independent of any such restriction.

Comparison with exact solution

The exact solution to this problem is given by Luikov and Mikhailov [1] on p. 255 and is

$$T = Ki_q\left[Fo - \tfrac{1}{6}(1 - 3X^2) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2\pi^2} \cos(n\pi X) \times \exp(-n^2\pi^2 Fo)\right] + \frac{\epsilon Lu Ko Ki_m}{Lu - 1} \times \left[Fo - FoLu + \sum_{n=1}^{\infty} (-1)^{n+1} \cos(n\pi X) \times \{\exp(-n^2\pi^2 Fo) - \exp(-n^2\pi^2 FoLu)\}\right] \quad (1.44)$$

$$\Theta = Ki_m[LuFo - \tfrac{1}{6}(1 - 3X^2) + \sum_{n=1}^{\infty} (-1)^{n+1} \cos(n\pi X) \exp(-n^2\pi^2 FoLu)]. \quad (1.45)$$

In the solutions (1.44) (1.45) the terms of the infinite series are very rapidly convergent and quasi-steady state solutions can be written as

$$T = Ki_q[Fo - \tfrac{1}{6}(1 - 3X^2)] - \epsilon Lu Ko Fo \quad (1.46)$$

$$\Theta = Ki_m[LuFo - \tfrac{1}{6}(1 - 3X^2)]. \quad (1.47)$$

The equations (1.46), (1.47) are exactly the same as (1.39) and (1.41) which are the approximate solutions found by the boundary-layer technique.

The relation (1.42), a solution found for the case ($\delta' < \delta$) when compared with (1.46) shows that Fo_3 has taken the place of $\frac{1}{6}$. As Fo_3 is a function of transfer coefficients this difference will therefore depend upon them. It has however been seen that $Fo_3 \gtrless \frac{1}{6}$ according as

$$Lu \gtrless \frac{1}{1 + \epsilon Ko(Ki_m/Ki_q)}$$

under which the approximate solution is found.

2. APPLICATION OF THE METHOD TO A PROBLEM WITH BOUNDARY CONDITIONS OF THE THIRD KIND

In this section we discuss the previous problem with boundary conditions of the third kind, and with the condition $\delta' > \delta$. The boundary conditions at the face $x = L$ will now be

$$-\lambda_q \frac{\partial t}{\partial x} + \alpha(t_c - t_0) - (1 - \epsilon)\rho q_m = 0 \qquad (2.1)$$

$$\lambda_m \frac{\partial \theta}{\partial x} + \lambda_m \delta_s \frac{\partial t}{\partial x} + q_m = 0 \qquad (2.2)$$

whereas the boundary conditions at $x = 0$ are the same as in the previous problem. As in the first phase we are neglecting thermal diffusion term, i.e. δ_s will be taken as zero whereas in the second phase δ_s will be kept in the transfer equations and the boundary conditions.

When $\delta_s = 0$, (2.2) becomes

$$\lambda_m \frac{\partial \theta}{\partial x} + q_m = 0. \qquad (2.3)$$

First phase

Proceeding exactly as before we obtain the two profiles as under:

$$t = t_0 + \frac{t_m - t_0}{\delta'[(2\lambda_q/\alpha) + \delta']}[\delta' - L + x]^2 \qquad (2.4)$$

and

$$\theta = \theta_0 - \frac{q_m}{2\lambda_m \delta}(\delta - L + x)^2 \qquad (2.5)$$

where

$$t_m = t_c - \frac{(1 - \epsilon)\rho q_m}{\alpha}. \qquad (2.6)$$

δ' in this problem is given by an implicit relation

$$\frac{[A'^2/B'^2 - (4\lambda_q^2/\alpha^2)]}{A'/B'} \log \frac{Z - A'/B'}{(2\lambda_q/\alpha) - A'/B'} + \frac{4\lambda_q^2/2\alpha}{A'/B'} \log \frac{Z\alpha}{2\lambda_q} + \left(Z - \frac{2\lambda_q}{\alpha}\right) = -B'\tau \qquad (2.7)$$

where

$$\left.\begin{aligned} Z &= \delta' + \frac{2\lambda_q}{\alpha} \\ A' &= 6K_{11}(t_m - t_0) \\ B' &= 3K_{12}\frac{q_m}{\lambda_m}. \end{aligned}\right\} \qquad (2.8)$$

The mass-transfer depth is again given by the same formula

$$\delta = \sqrt{(6K_{21}\tau)}. \qquad (2.9)$$

The transition time τ_1' (for heat transfer when $\delta' > \delta$) can be found from (2.7) by putting $\delta' = L$. The transition time for mass transfer is from (2.9)

$$\tau_2' = \frac{L^2}{6K_{21}}. \qquad (2.10)$$

Second phase

The initial distributions for the second phase are

$$t = t_0 + \frac{t_m - t_0}{L[(2\lambda_q/\alpha) + L]}x^2, \qquad \tau = \tau_1' \qquad (2.11)$$

$$\theta = \theta_0 - \frac{q_m}{2\lambda_m L}x^2, \qquad \tau = \tau_2'. \qquad (2.11\text{A})$$

Here we include the effect of thermal diffusion, i.e. δ_s is retained in both the transfer equations and boundary conditions. Hence we consider

$$\frac{\partial t}{\partial \tau} = K_{11}\frac{\partial^2 t}{\partial x^2} + K_{12}\frac{\partial^2 \theta}{\partial x^2} \tag{2.12}$$

$$\frac{\partial \theta}{\partial \tau} = K_{21}\frac{\partial^2 \theta}{\partial x^2} + K_{22}\frac{\partial^2 t}{\partial x^2}. \tag{2.13}$$

With boundary conditions given by equations (2.1) and (2.2) and where, now

$$K_{11} = a_q + \frac{\epsilon\rho c_m a_m \delta_s}{c_q} \tag{2.14}$$

$$K_{12} = \frac{\epsilon\rho}{c_q} c_m a_m \tag{2.15}$$

$$K_{21} = a_m \tag{2.16}$$

$$K_{22} = a_m \delta_s. \tag{2.17}$$

In this case too we assume parabolic profiles for temperature and mass-transfer potential. Proceeding exactly as before we get temperature and mass-transfer potential in the non-dimensional form as

$$\begin{aligned} T = \frac{t - t_0}{t_c - t_0} = &\left[1 - (1-\epsilon)\, LuKo\frac{Ki_m}{Bi_q}\right] - \tfrac{1}{2}\epsilon LuKoKi_m\left[1 - X^2 + \frac{2}{Bi_q}\right] \\ &+ \left[\frac{1 - (1-\epsilon)\, LuKoKi_m/Bi_q}{1 + 2/Bi_q} - \tfrac{1}{2}\epsilon LuKoKi_m\right]\left[X^2 - 1 - \frac{2}{Bi_q}\right] \\ &\times \left[\exp\left\{-\frac{3Bi_q}{3 + Bi_q}(Fo - Fo_1')\right\}\right] \end{aligned} \tag{2.18}$$

where

$$Fo_1' = a_q\tau_1'/L^2 \tag{2.19}$$

and

$$\begin{aligned} \Theta = \frac{\theta_0 - \theta}{\theta_0 - \theta_p} = &\, Ki_m[LuFo - \tfrac{1}{6}(1 - 3X^2)] \\ &- Pn(X^2 - \tfrac{1}{3})\left[\frac{Bi_q - (1-\epsilon)LuKi_mKo}{Bi_q + 2} - \tfrac{1}{2}\epsilon LuKoKi_m\right]\left[\exp\left\{-\frac{3Bi_q}{3 + Bi_q}\right.\right. \\ &\left.\left.\times (Fo_2' - Fo_1')\right\} - \exp\left\{-\frac{3Bi_q}{3 + Bi_q} \times (Fo - Fo_1')\right\}\right] \end{aligned} \tag{2.20}$$

where

$$Fo_2' = a_q\tau_2'/L^2. \tag{2.21}$$

Comparison with exact solution

The solutions (2.18), (2.20) for large values of time can be written as

$$T = \left[1 - (1-\epsilon)LuKo\frac{Ki_m}{Bi_q}\right] - \tfrac{1}{2}\epsilon LuKoKi_m\left[1 - X^2 + \frac{2}{Bi_q}\right] \tag{2.22}$$

and

$$\begin{aligned} \Theta = &\, Ki_m[LuFo - \tfrac{1}{6}(1 - 3X^2)] \\ &- Pn\left[\frac{Bi_q - (1-\epsilon)LuKoKi_m}{Bi_q + 2} - \tfrac{1}{2}\epsilon LuKoKi_m\right] \\ &\times \left[(X^2 - \tfrac{1}{3})\exp\left\{-\frac{3Bi_q}{3 + Bi_q} \times (Fo_2' - Fo_1')\right\}\right]. \end{aligned} \tag{2.23}$$

The exact solution to this problem for quasi-steady state is given by Luikov and Mikhailov [1] on p. 282 as

$$T = \left[1 - (1-\epsilon)LuKo\frac{Ki_m}{Bi_q}\right] - \tfrac{1}{2}\epsilon LuKoKi_m\left[1 - X^2 + \frac{2}{Bi_q}\right] \tag{2.24}$$

$$\Theta = Ki_m[LuFo - \tfrac{1}{6}(1 + \epsilon LuKoPn)(1 - 3X^2)]. \tag{2.25}$$

Comparing the exact and approximate solutions we observe that the expression for temperature obtained by the approximate method is the same as that given by the exact method where as the non-dimensional mass-transfer potential (2.22) given by the approximate method differs from the exact value (2.27) in the term containing Pn. This may be due to our neglecting the thermal diffusion term in the first phase. If however the thermal diffusion term is neglected in both the phases the above discrepancy in case of mass-transfer potential results also vanishes.

3. A NON-LINEAR PROBLEM

In this section we discuss heat and mass transfer in a porous infinite plate with the boundary conditions of first kind, assuming the heat conductivity and the mass conductivity to be linear functions of temperature and mass-transfer potential.

The boundary conditions in this case at $x = L$ are

$$\theta(L, \tau) = \theta_1 \tag{3.1}$$

$$t(L, \tau) = t_1. \tag{3.2}$$

The boundary conditions at $x = 0$ are the same as in the previous problems.

The diffusivities of heat and mass can be written as linear functions of temperature and mass-transfer potential because both the specific heat capacity and specific mass capacity as well as the density of the medium are assumed to be constant. Thus all other parameters are constant except the diffusivities.

The differential equations for the process can be written as

$$\frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x}\left(a_q \frac{\partial t}{\partial x}\right) + \frac{\epsilon\rho}{c_q} c_m \frac{\partial}{\partial x}\left(a_m \frac{\partial \theta}{\partial x}\right) \tag{3.3}$$

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(a_m \frac{\partial \theta}{\partial x}\right), \quad 0 < x < L \qquad \tau > 0. \tag{3.4}$$

The diffusivities are considered to be of the following form

$$\left.\begin{aligned} a_q &= a_q^0(1 + \lambda_1\theta + \lambda_2 t) \quad (3.5)\\ a_m &= a_m^0(1 + \mu_1\theta + \mu_2 t) \end{aligned}\right| \text{for the } \delta' > \delta \text{ case.} \tag{3.6}$$

Case 1 : $\delta' > \delta$. Adopting the same procedure as in the earlier two cases we have the following two thicknesses of heat and mass layers.

$$\delta'(\tau) = \left[-\frac{H}{2} + \sqrt{\left(\frac{H^2}{4} + A''\right)}\right]\sqrt{\tau} \tag{3.7}$$

$$\delta(\tau) = \sqrt{\{12a_m^0(1 + \lambda_1\theta_1 + \lambda_2 t_1)\tau\}} \tag{3.8}$$

where

$$H = \sqrt{\{12a_m^0(1 + \lambda_1\theta_1 + \lambda_2 t_1)\}}\frac{\epsilon\rho c_m(\theta - \theta_1)}{c_q(t_0 - t_1)} \tag{3.9}$$

$$A'' = 12a_q^0(1 + \mu_1\theta_1 + \mu_2 t_1). \tag{3.10}$$

The temperature and mass-transfer potential distributions in the first phase are

$$t = t_0 + \frac{t_1 - t_0}{\delta'^2}(\delta' - L + x)^2 \tag{3.11}$$

$$\theta = \theta_0 + \frac{\theta_1 - \theta_0}{\delta^2}(\delta - L + x)^2. \tag{3.12}$$

The initial distribution of temperature and mass-transfer potential for the second phase are

$$t = t_0 + \frac{t_1 - t_0}{L^2}x^2, \quad \tau = \tau_1'' \tag{3.13}$$

$$\theta = \theta_0 + \frac{\theta_1 - \theta_0}{L^2}x^2, \quad \tau = \tau_2''. \tag{3.14}$$

As already stated, in the second phase we shall consider the equations in full, i.e. thermal diffusion term will be taken into account and in that case the equations are

$$\frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x}\left(K_{11}\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial x}\left(K_{12}\frac{\partial \theta}{\partial x}\right) \tag{3.15}$$

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(K_{21}\frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial x}\left(K_{22}\frac{\partial t}{\partial x}\right) \tag{3.16}$$

where K_{11}, K_{12}, K_{21} and K_{22} are given by equations (2.14) to (2.17).

Assuming the profiles in the same form as in equations (1.33), (1.34) and following the same procedure as before we get two simultaneous differential equations for the determination of A'_2 and B'_2 as

$$-\frac{L^2}{3}\frac{\mathrm{d}A'_2}{\mathrm{d}\tau} = K^0_{11}A'_2 + K^0_{12}B'_2 \tag{3.17}$$

$$-\frac{L^2}{3}\frac{\mathrm{d}B'_2}{\mathrm{d}\tau} K^0_{21}B'_2 + K^0_{22}A'_2 \tag{3.18}$$

where K^0_{11}, K^0_{12}, K^0_{21}, K^0_{22} are the values of K_{11}, K_{12}, K_{21}, K_{22} respectively at $x = L$. Solving (3.17) and (3.18) taking into account the initial distribution (3.13) (3.14) we get the values of A'_2, B'_2 which on substitution in the profiles determine the distributions as

$$T = 1 + \frac{A'_2 L^2}{t_1 - t_0}(X^2 - 1) \tag{3.19}$$

and

$$\Theta = 1 + \frac{B'_2 L^2}{\theta_1 - \theta_0}(X^2 - 1) \tag{3.20}$$

where

$$A'_2 = C_1 \exp\left(-\frac{3}{L^2}K^0_{21}v'^2_1\tau\right) + C_2 \exp\left(-\frac{3}{L^2}K^0_{21}v'^2_1\tau\right) \tag{3.21}$$

$$\frac{C_1 L^2}{t_1 - t_0} = \frac{1}{\Delta}\left[\left(v'^2_2 - \frac{K^0_{11}}{K^0_{21}}\right)\exp\left(-\frac{3}{L^2}K^0_{21}v'^2_2\tau''_2\right) + \frac{\theta_0 - \theta_1}{t_1 - t_0}\frac{K^0_{12}}{K^0_{21}}\exp\left(-\frac{3}{L^2}K^0_{22}v'^2_2\tau''_1\right)\right] \tag{3.22}$$

$$\frac{C_2 L^2}{t_1 - t_0} = -\frac{1}{\Delta}\left[\left(v'^2_1 - \frac{K^0_{11}}{K^0_{21}}\right)\exp\left(-\frac{3}{L^2}K^0_{21}v'^2_1\tau''_2\right) + \frac{\theta_0 - \theta_1}{t_1 - t_0}\exp\left(-\frac{3}{L^2}K^0_{21}v'^2_1\tau''_1\right)\right] \tag{3.23}$$

$$\Delta = \left(v'^2_2 - \frac{K^0_{11}}{K^0_{21}}\right)\exp\left\{-\frac{3}{L^2}(v'^2_1\tau''_1 + v'^2_2\tau''_2)\right\} - \left(v'^2_1 - \frac{K^0_{11}}{K^0_{21}}\right)\exp\left\{-\frac{3}{L^2}K^0_{21}(v'^2_2\tau''_1 + v'^2_1\tau''_2)\right\} \tag{3.24}$$

and

$$v'^2_1 = \frac{1}{2}\left[\left(1 + \frac{1}{Lu^*} + \epsilon KoPn\right) - \sqrt{\left\{\left(1 + \frac{1}{Lu^*} + \epsilon KoPn\right)^2 - \frac{4}{Lu^*}\right\}}\right] \tag{3.25}$$

$$v'^2_2 = \frac{1}{2}\left[\left(1 + \frac{1}{Lu^*} + \epsilon KoPn\right) + \sqrt{\left\{\left(1 + \frac{1}{Lu^*} + \epsilon KoPn\right)^2 - \frac{4}{Lu^*}\right\}}\right] \tag{3.26}$$

also

$$Lu^* = \frac{a^0_m(1 + \lambda_1\theta_1 + \lambda_2 t_1)}{a^0_q(1 + \mu_1\theta_1 + \mu_2 t_1)} = \frac{Lu_0(1 + \lambda'_1\Theta_1 + \lambda'_2 T_1)}{(1 + \mu'_1\Theta_1 + \mu'_2 T_1)} \tag{3.27}$$

where

$$\Theta_1 = \frac{\theta_1}{\theta_0 - \theta_1}, \qquad T_1 = \frac{t_1}{t_1 - t_0},$$

$$\lambda'_1 = \lambda_1(\theta_0 - \theta_1), \qquad \lambda'_2 = \lambda_2(t_1 - t_0),$$

$$\mu'_1 = \mu_1(\theta_0 - \theta_1), \qquad \mu'_2 = \mu_2(t_1 - t_0).$$

The quantities K^0_{11}/K^0_{21} and K^0_{12}/K^0_{21} in the above equations can be written in the non-dimensional form as

$$\frac{K^0_{11}}{K^0_{21}} = \frac{1}{Lu^*} + \epsilon KoPn \tag{3.28}$$

$$\frac{K^0_{12}}{K^0_{21}}\,\frac{\theta_0 - \theta_1}{t_1 - t_0} = \epsilon Ko. \tag{3.29}$$

The quantities $(3/L^2)K^0_{21}\tau''_1$ and $(3/L^2)K^0_{21}\tau''_2$ occurring in the above equations are non-dimensional. Also in equation (3.20)

$$B'_2 = D_1\left[\frac{v'^2_1 - \dfrac{K^0_{11}}{K^0_{21}}}{K^0_{11}/K^0_{21}}\right]\exp\left(-\frac{3}{L^2}K^0_{21}\tau v'^2_1\right) + D_2\left[\frac{v'^2_2 - K^0_{11}/K^0_{21}}{K^0_{12}/K^0_{21}}\right]\exp\left(-\frac{3}{L^2}K^0_{21}\tau v'^2_2\right) \tag{3.30}$$

where

$$\frac{D_1L^2}{\theta_1-\theta_0}=-\frac{t_1-t_0}{\Delta(\theta_0-\theta_1)}\left[\frac{K_{12}^0}{K_{21}^0}\frac{\theta_0-\theta_1}{t_1-t_0}\exp\left(-\frac{3}{L^2}K_{21}^0\tau_1''\nu_2'^2\right)+\left(\nu_2'^2-\frac{K_{11}^0}{K_{21}^0}\right)\exp\left(-\frac{3}{L^2}K_{21}^0\tau_2''\nu_2'^2\right)\right] \tag{3.31}$$

$$\frac{D_2L^2}{\theta_1-\theta_0}=\frac{t_1-t_0}{\Delta(\theta_0-\theta_1)}\left[\frac{K_{12}^0}{K_{21}^0}\frac{\theta_0-\theta_1}{t_1-t_0}\exp\left(-\frac{3}{L^2}K_{21}^0\tau_1''\nu_1'^2\right)+\left(\nu_1^2-\frac{K_{11}^0}{K_{21}^0}\right)\exp\left(-\frac{3}{L^2}K_{21}^0\tau_2''\nu_2'^2\right)\right] \tag{3.32}$$

Case 2: $\delta' < \delta$. In this case in the first phase we assume the mass diffusivity to be a linear function of only the temperature whereas the diffusivity of heat follows the same linear law as expressed by (3.5). The equation (3.6) in this case is written as

$$a_m = a_m^0(1+\lambda_2 t). \tag{3.33}$$

Proceeding exactly in the same manner as before we find

$$\delta'(\tau)=\frac{H_1}{2(H_2+1)}\left[-1+\sqrt{\left\{1+\frac{4D'(H_2+1)}{H_1^2}\right\}}\right]\sqrt{\tau} \tag{3.34}$$

and

$$\delta(\tau)=\sqrt{\{12a_m^0(1+\lambda_2 t_1)\tau\}} \tag{3.35}$$

where

$$D' = 12a_q^0(1+\mu_1\theta_1+\mu_2 t_1) \tag{3.36}$$

$$H_1 = \lambda_2'\epsilon Ko\sqrt{\left(\frac{12a_m^0}{1+\lambda_2 t_1}\right)} \tag{3.37}$$

$$H_2 = \epsilon Ko\left(\frac{1+\lambda t_0}{1+\lambda_2 t_1}\right) \tag{3.38}$$

$$\lambda_2' = \lambda_2(t_1-t_0) \tag{3.39}$$

and

$$Ko = \frac{\rho c_m}{c_q}\frac{\theta_0-\theta_1}{t_1-t_0}. \tag{3.40}$$

The transition times τ_1''', τ_2''' for heat transfer and mass transfer respectively can be found from (3.34) and (3.35) by putting $\delta = \delta' = L$.

The temperature and mass-transfer potential distribution after the second phase, in this case shall be similar to (3.19), (3.20) except that λ_1 has to be made equal to zero and τ_1'', τ_2'' have to be replaced by τ_1''', τ_2''' respectively.

The inequalities $\delta' \gtrless \delta$ in this problem take the forms respectively as

$$Lu^* \gtrless \frac{1}{1+\epsilon Ko}. \tag{3.41}$$

Solution (3.19) corresponds to the upper inequality and (3.20) to the lower one.

Some numerical results

The results for this problem have been exhibited graphically in Fig. 1 for

$$Lu^* < \frac{1}{1+\epsilon Ko}$$

and in Figs. 2 and 3 for

$$Lu^* > \frac{1}{1+\epsilon Ko}.$$

In Fig. 1 non-dimensional temperature T has been plotted against non-dimensional time Fo for $\mu_1' = \mu_2' = \lambda_2' = 0$ and $\Theta_1 = 1$ for different values of λ_1', i.e. this is the graph of T vs. Fo when Lu^* is a linear function of mass-transfer potential.

Figure 2 is the graph of T vs. Fo for the above stated values for the situation

$$Lu^* > \frac{1}{1+\epsilon Ko}.$$

In Fig. 3 non-dimensional mass-transfer

potential Θ has been plotted against non-dimensional time Fo for $\mu_1' = \mu_2' = \lambda_1' = 0$ and $T_1 = 1$, i.e. when Lu is a linear function of temperature.

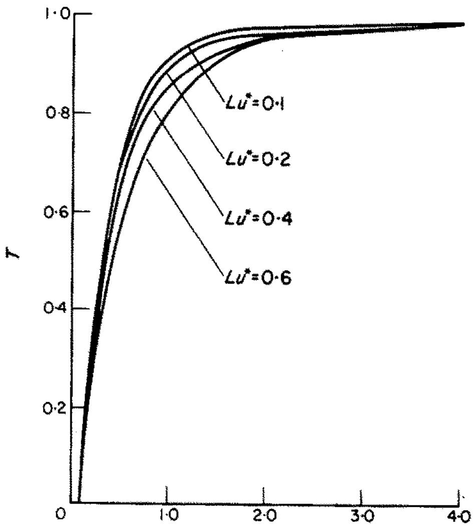

FIG. 1. T vs. Fo for different values of Lu^*. $Lu_0 = 0{\cdot}1$; $\epsilon = 0{\cdot}5$; $Ko = 1{\cdot}2$ and $Pn = 0{\cdot}5$.

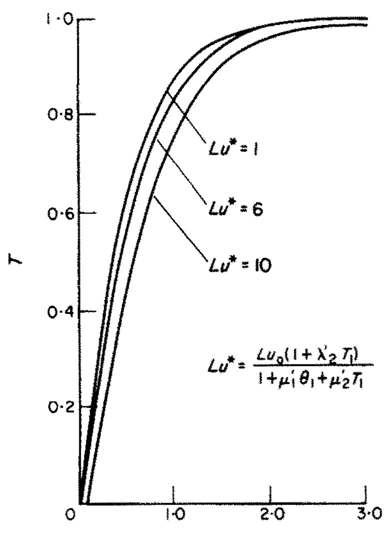

FIG. 2. T vs. Fo for different values of Lu^*. $Lu_0 = 0{\cdot}1$; $\epsilon = 0{\cdot}5$; $Ko = 1{\cdot}2$ and $Pn = 0{\cdot}5$.

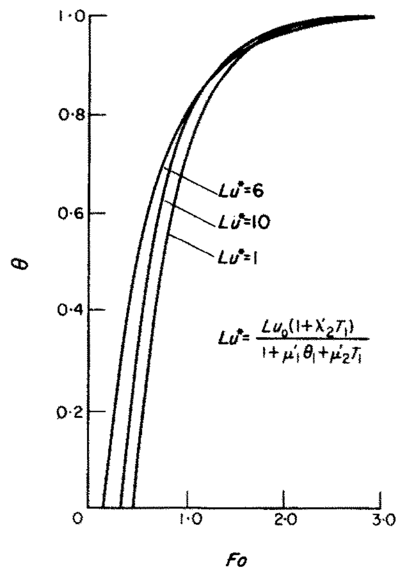

FIG. 3. Θ vs. Fo for different values of Lu^*. $Lu_0 = 0{\cdot}1$; $\epsilon = 0{\cdot}5$; $Ko = 1{\cdot}2$ and $Pn = 0{\cdot}5$.

CONCLUSION

In this paper we have tried to establish an approximate integral method for the solution of coupled equations of heat and mass transfer in a porous medium. It has been shown that the results obtained by this technique for certain linear problems are quite close to the ones obtained by elaborate exact procedures. The method discussed here also provides quick approximate results to non-linear problems with temperature and moisture dependent physical properties of the medium. With slight modifications the method can be extended to the solution of heat- and mass-transfer equations for binary gas mixtures and equations of molar and molecular heat and mass transfer.

The extension of the idea of the penetration depth (Biot and Goodman) to this case necessitates the knowledge of the relative rates of the process of heat and mass transfer, thus giving a relation between the characteristics of the processes. This condition turns out to be slightly different from Luikov's criterion for the corresponding situations. It is noted that the condition is dependent on the type of boundary conditions assumed at the surface.

Finally, results for heat and mass transfer with a_m and a_q assumed to be linear function of temperature and mass-transfer potential have been obtained. Results, where the diffusivities of the medium vary as any other power of temperature and mass transfer potential, can be obtained by a similar procedure.

The method has certain limitations. It has not been possible to obtain analytic results in the case where δ_s is taken into account from the start of the process, but exact equations can be considered in the second stage, i.e. after the transient time. The technique would therefore be specially useful for analysis of the heat- and mass-transfer problems in thin plates where the transition time is very small and the omission of thermal diffusion coefficient in the first stage does not materially effect the results. On the same account the method is quite useful in processes where thermal diffusion is very small. Further, the method cannot be applied to problems with initially non-uniform distributions of temperature and moisture transfer potential as the concept of penetration depths for both the heat and moisture cannot be properly explained in the case of non-uniform initial conditions.

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Résumé—La technique de la couche limite développée par Goodman dans les problèmes de transport de chaleur a été étendue aux phénomènes avec couplage de transport de chaleur et de masse dans les milieux poreux. Pour justifier l'application de la méthode aux problèmes de transport de chaleur et de masse, un problème linéaire avec des conditions aux limites de seconde et de troisième espèce respectivement a été discuté et les résultats comparés avec des solutions exactes déjà connues. Un problème non-linéaire où l'on a supposé le nombre de Luikov dépendant linéairement de la température et du potentiel de transport de masse a été également discuté et les résultats ont été représentés graphiquement.

Zusammenfassung—Die von Goodman für Wärmeübergangsprobleme entwickelte Grenzschichttechnik wurde auf gekoppelte Phänomene des Wärme- und Stoffüberganges in porösen Medien erweitert. Um die Anwendung der Methode auf Wärme- und Stoffübergangsprobleme zu rechtfertigen, wurde ein lineares Problem mit Randbedingungen zweiter, bzw. dritter Art untersucht und die Ergebnisse mit bereits bekannten exakten Lösungen verglichen. Ein nichtlineares Problem, für das die Luikov-Zahl als linear abhängig von der Temperatur und dem Stoffübergangspotential angenommen wurde, ist ebenfalls betrachtet worden und einige Ergebnisse sind grafisch dargestellt.

Аннотация—Техника пограничного слоя, развитая Т. Р. Гудманом в задачах теплопереноса, обобщена на явления совместного тепло-и массопереноса в пористой среде. Для обоснования применимости метода к задачам тепло-и массопереноса рассматривается линейная задача с граничными условиями второго и третьего рода, соответственно. Полученные результаты сравнивались с уже известными точными решениями. Рассматривалась также нелинейная задача, в которой число Лыкова, линейно зависит от температуры и потенциала массопереноса. Некоторые результаты представлены графически.